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$[(2 \times 7) + 1][(4 \times 7)/(4^2 + 7^2)] = 13 \times 15 \times \frac{2}{3} \frac{4}{7} = 3 \times 4 \times 7 = 84$  ;  $[(2 \times 26) - 1][(2 \times 26 + 1)]$   
 $[(15 \times 26)/(15^2 + 26^2)] = 51 \times 53 \times \frac{2}{3} \frac{4}{7} = 3 \times 15 \times 26 = 1170$  ;  $[(2 \times 97) - 1][(2 \times 97 + 1)]$   
 $[(56 \times 97)/(56^2 + 97^2)] = 193 \times 195 \times \frac{5}{12} \frac{4}{7} \frac{2}{5} = 3 \times 56 \times 97 = 16296$  ;  $[(2 \times 362) - 1][(2 \times 362) + 1]$   
 $[(209 \times 362)/(209^2 + 362^2)] = 723 \times 725 \times \frac{7}{14} \frac{5}{7} \frac{2}{5} = 3 \times 209 \times 362 = 226974$  ;  
 etc.

Here it should be noticed that in canceling both sides the denominator of the half-sine disappears, and three times the product of the terms of the  $n$ th even convergent in the expansion of  $1/3$  brings the area to light ; also observe, since sides and denominator fall out of view, and factor 3 stands constant, the area must be determined by the numerator of these half-sines, and *this* series may be continued by use of Magic  $M=14$  ;  $14 \times 2 = 28$  ;  $(14 \times 28) - 2 = 390$  ;  $(14 \times 390) - 28 = 5432$  ;  $(14 \times 5432) - 390 = 75658$  ; etc.

Also solved by *JOSIAH H. DRUMMOND*, *M. A. GRUBER*, and *G. B. M. ZERE*.

### AVERAGE AND PROBABILITY.

60. Proposed by *B. F. FINKEL*, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Four points are taken at random within an ellipse. What is the chance that they form a reentrant quadrilateral ?

Solution by *G. B. M. ZERE*, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

We will solve this problem for the quadrant, the semi-ellipse, and the whole ellipse.

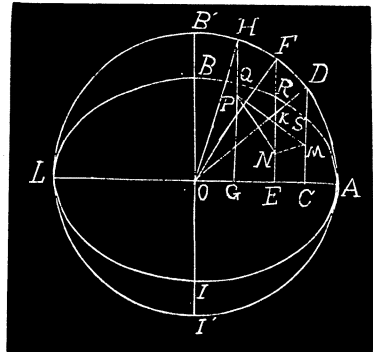
Let  $ABLI$  be the ellipse, and  $ABLI'$  the circumscribing circle ;  $M, N, P$  the three random points ; through  $M, N, P$  draw  $CD, EF, GH$  perpendicular to  $AO$ ,  $EF$  intersecting  $MP$  at  $K$ . The triangle will pass through all the possible variations by considering only those relative positions of the points in which  $CD$  lies to the right of  $GH$ , and  $EF$  between  $CD$  and  $GH$ .

If the fourth point falls anywhere on the triangle formed by joining the points  $M, N, P$ , the quadrilateral thus formed will be reentrant.

Let  $OA=a$ ,  $OB=b$ ,  $GP=x$ ,  $CM=y$ ,  $EN=z$ ,  $GQ=x'$ ,  $CS=y'$ ,  $ER=z'$ ,  $EK=z''$ ,  $\angle GOH=\theta$ ,  $\angle COD=\varphi$ ,  $\angle EOF=\psi$ .

Then we have  $x'=b\sin\theta$ ,  $y'=b\sin\varphi$ ,  $z'=b\sin\psi$ ,  $v=1/(\cos\varphi-\cos\theta)$ ,  $z''=v[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)]$ .

Area  $MNP = \frac{1}{2}a[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)+z(\cos\theta-\cos\varphi)] = u$ , when  $z < z''$ . Area  $MNP = \frac{1}{2}a[x(\cos\psi-\cos\varphi)+y(\cos\theta-\cos\psi)+z(\cos\varphi-\cos\theta)] = u_1$ , when



$z > z''$ . An element of surface at  $M$  is  $a \sin \varphi d\varphi dy$ , at  $N$  it is  $a \sin \psi d\psi dz$ , at  $P$  it is  $a \sin \theta d\theta dx$ .

The limits of  $\theta$  are (for quadrant) 0 and  $\frac{1}{2}\pi$ ; of  $\varphi$ , 0 and  $\theta$ ; of  $\psi$ ,  $\varphi$  and  $\theta$ ; of  $x$ , 0 and  $x'$ ; of  $y$ , 0 and  $y'$ ; of  $z$ , 0 and  $z''$ , and  $z'$ .

Hence the required average area is,

$$\begin{aligned}
 \Delta &= \frac{\int_0^{\frac{1}{2}\pi} \int_0^\theta \int_\varphi^\theta \int_0^{x'} \int_0^{y'} \left( \int_0^{z''} u dz + \int_{z''}^{z'} u_1 dz \right) a \sin \theta d\theta a \sin \varphi d\varphi a \sin \psi d\psi dx dy}{\int_0^{\frac{1}{2}\pi} \int_0^\theta \int_\varphi^\theta \int_0^{x'} \int_0^{y'} \int_0^{z'} a \sin \theta d\theta a \sin \varphi d\varphi a \sin \psi d\psi dx dy dz} \\
 &= \frac{384}{\pi^3 b^3} \int_0^{\frac{1}{2}\pi} \int_0^\theta \int_\varphi^\theta \int_0^{x'} \int_0^{y'} \left( \int_0^{z''} u dz + \int_{z''}^{z'} u_1 dz \right) \sin \theta \sin \varphi \sin \psi d\theta d\varphi d\psi dx dy \\
 &= \frac{96a}{\pi^3 b^3} \int_0^{\frac{1}{2}\pi} \int_0^\theta \int_\varphi^\theta \int_0^{x'} \int_0^{y'} \{ [x(\cos \varphi - \cos \psi) + y(\cos \psi - \cos \theta)]^2 + [x(\cos \varphi - \cos \psi) \\
 &\quad + y(\cos \psi - \cos \theta) + b \sin \psi (\cos \theta - \cos \varphi)]^2 \} \sin \theta \sin \varphi \sin \psi d\theta d\varphi d\psi dx dy \\
 &= \frac{32a}{\pi^3 b^3} \int_0^{\frac{1}{2}\pi} \int_0^\theta \int_\varphi^\theta \int_0^{x'} [6x^2 \sin \varphi (\cos \varphi - \cos \psi)^2 + 6bx \sin^2 \varphi (\cos \varphi - \cos \psi)(\cos \psi \\
 &\quad - \cos \theta) + 6bx \sin \varphi \sin \psi (\cos \varphi - \cos \psi)(\cos \theta - \cos \varphi) + 2b^2 \sin^3 \varphi (\cos \psi - \cos \theta)^2 \\
 &\quad + 3b^2 \sin \varphi \sin^2 \psi (\cos \theta - \cos \varphi)^2 + 3b^2 \sin^2 \varphi \sin \psi (\cos \theta - \cos \varphi)(\cos \psi - \cos \theta)] \\
 &\quad \times \sin \theta \sin \varphi \sin \psi d\theta d\varphi d\psi dx. \\
 \Delta &= \frac{32ab}{\pi^3} \int_0^{\frac{1}{2}\pi} \int_0^\theta \int_\varphi^\theta [2 \sin^3 \theta \sin \varphi (\cos \varphi - \cos \psi)^2 + 2 \sin \theta \sin^3 \varphi (\cos \psi - \cos \theta)^2 \\
 &\quad + 3 \sin^2 \theta \sin^2 \varphi (\cos \varphi - \cos \psi)(\cos \psi - \cos \theta) + 3 \sin \theta \sin \varphi \sin^2 \psi (\cos \theta - \cos \varphi)^2 \\
 &\quad + 3 \sin^2 \theta \sin \varphi \sin \psi (\cos \varphi - \cos \psi)(\cos \theta - \cos \varphi) \\
 &\quad + 3 \sin \theta \sin^2 \varphi \sin \psi (\cos \psi - \cos \theta)(\cos \theta - \cos \varphi)] \sin \theta \sin \varphi \sin \psi d\theta d\varphi d\psi \\
 &= \frac{16ab}{\pi^3} \int_0^{\frac{1}{2}\pi} \int_0^\theta [4 \sin^2 \theta \cos^2 \varphi + 4 \sin^2 \varphi \cos^2 \theta + 4 \sin^2 \theta \cos^2 \theta + 4 \sin^2 \varphi \cos^2 \theta \\
 &\quad + \sin^2 \theta \cos \theta \cos \varphi + \sin^2 \varphi \cos \varphi \cos \theta - 6 \sin \theta \cos \theta \sin \varphi \cos \varphi]
 \end{aligned}$$

$$\begin{aligned}
& + 6\cos^3\theta\cos\varphi + 6\cos\theta\cos^3\varphi + 12 + 6\cos^2\theta + 6\cos^2\varphi - 36\cos\theta\cos\varphi \\
& - 12\sin\theta\sin\varphi - 9(\theta - \phi)\sin\theta\cos\varphi + 9(\theta - \phi)\sin\varphi\cos\theta] \sin^2\theta\sin^2\varphi d\theta d\varphi \\
& = \frac{8ab}{9\pi^3} \int_0^{4\pi} (69\theta + 36\theta\cos\theta - 12\theta\sin^2\theta - 12\theta\sin^4\theta - 60\sin\theta - 45\sin\theta\cos\theta \\
& - 10\sin^3\theta\cos\theta + 3\sin^5\theta\cos\theta)\sin^2\theta d\theta \\
& = \frac{ab}{\pi} \left( \frac{35}{12} + \frac{16}{3\pi} - \frac{131}{3\pi^2} \right).
\end{aligned}$$

For the semi-ellipse above the major axis, the limits of  $\theta$  are 0 and  $\pi$ , and those of the other variables the same as above. The number of ways the three points can be taken in the semi-ellipse is eight times the number of ways in a quadrant, and hence we get

$$\begin{aligned}
\Delta_1 & = \frac{ab}{9\pi^3} \int_0^\pi (69\theta + 36\theta\cos\theta - 12\theta\sin^2\theta - 12\theta\sin^4\theta - 60\sin\theta - 45\sin\theta\cos\theta \\
& - 10\sin^3\theta\cos\theta + 3\sin^5\theta\cos\theta)\sin^2\theta d\theta = \frac{ab}{\pi} \left( \frac{35}{24} - \frac{32}{3\pi^2} \right).
\end{aligned}$$

For the limits of  $\theta$  are 0 and  $2\pi$ , and the points can be taken eight times the number of ways in semi-ellipse. Hence

$$\begin{aligned}
\Delta_2 & = \frac{ab}{72\pi^3} \int_0^{2\pi} (69\theta + 36\theta\cos\theta - 12\theta\sin^2\theta - 12\theta\sin^4\theta - 60\sin\theta - 45\sin\theta\cos\theta \\
& - 10\sin^3\theta\cos\theta + 3\sin^5\theta\cos\theta)\sin^2\theta d\theta = 35ab/48\pi.
\end{aligned}$$

Let  $C$ ,  $C_1$ ,  $C_2$  be the respective chances required.

$$C = \frac{4\Delta}{\pi ab} = \frac{4}{\pi^2} \left( \frac{35}{12} + \frac{16}{3\pi} - \frac{131}{3\pi^2} \right); \quad C_1 = \frac{4\Delta_1}{\pi ab} = \left( \frac{35}{42} - \frac{32}{3\pi^2} \right);$$

$$C_2 = \frac{4\Delta_2}{\pi ab} = \frac{35}{12\pi^2}.$$